It should also be pointed out that in order to evaluate the effect of additional modes on the stability requirements, two modes of antisymmetric elastic motion were also considered. However, the inclusion of these additional modes did not appreciably alter the stability criteria. The effect of relaxing the constrained orbit assumption was also investigated. The E_1,E_3 stability criteria remained unaffected, while the E_2,E_4 requirements were only slightly modified.

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Transonic Nozzle Flow with Nonuniform Total Energy

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Introduction

THE analysis of the performance of a convergent-divergent nozzle usually relies on a characteristic solution of the supersonic flowfield. Therefore, the transonic flow within the throat region must be known with sufficient accuracy to permit the development of a supersonic start line from which the generation of the characteristic net can be started. Calculating the transonic flow region for a potential flow presents no difficulty since this problem has received much attention and the results, notably those of Sauer[1] and Hall,[2] are well known. More recent studies[3–5] have also treated the flow in the transonic region; however, these too have been restricted to potential flow. There are instances in which a flow may have a rotational component, for example in a rocket nozzle with nonuniform total energy across the flow. In such cases, it has been a common practice to use the potential results to determine the start line in the absence of a solution for a rotational flow.

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The purpose of this note is to present a solution for transonic flow in a nozzle throat with a variable stagnation speed of sound. The procedure used to solve the equations is similar to that used by Hall. However, the governing equations are written with the stream function as an independent variable to conveniently accommodate the variation of stagnation conditions across the flow.

Analysis

The origin of the cylindrical coordinate system (r,x) for the transonic region in an axisymmetric convergent-divergent nozzle will be taken at the geometric throat. The throat wall contour can be generally expressed as being circular, parabolic or hyperbolic. It is assumed that the variation in total temperature is produced upstream of the throat section. Then, neglecting transport phenomena, the flow can be considered isentropic along each streamline. However, the stagnation temperature (or stagnation speed of sound) is taken to depend upon the stream function. In addition, a uniform composition and a constant total pressure is assumed to exist throughout the flowfield.

It is then convenient to use $\xi = x/L$ and $\eta = \psi/\psi_w$ as independent variables where L is a characteristic axial length which will be defined later and $\psi_w = \dot{m}/2\pi$ is the value of the stream function corresponding to the wall of the nozzle. The transformation of the governing equations from x,r coordinates to ξ,η coordinates is accomplished with

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{L}\frac{\partial}{\partial \xi} - \frac{2\pi r}{\dot{m}}\rho v\frac{\partial}{\partial \eta} \quad (1)$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{2\pi r}{\dot{m}}\rho u\frac{\partial}{\partial \eta} \quad (2)$$

With the above transformation formulas, the continuity and Euler equations for axially symmetric compressible flow become

$$\frac{\partial(\rho u)}{\partial \xi} + \frac{2\pi r}{\dot{m}}\rho^2 L\left(u\frac{\partial v}{\partial \eta} - v\frac{\partial \mu}{\partial \eta}\right) + \rho\frac{vL}{r} = 0 \quad (3)$$

$$\rho u \partial u/\partial\xi = -(\partial p/\partial\xi) + (2\pi r/\dot{m})\rho v L \partial p/\partial\eta \quad (4)$$

$$\partial v/\partial\xi = -(2\pi r/\dot{m})L\partial p/\partial\eta \quad (5)$$

Eliminating $\partial p/\partial\eta$ between Eqs. (4) and (5) yields

$$\rho u \partial u/\partial\xi + \rho v \partial v/\partial\xi = -(\partial p/\partial\xi) \quad (6)$$

which if integrated would lead to the compressible Bernoulli equation along a streamline. Since the flow is isentropic along a streamline, we can write

$$\partial\rho/\partial\xi = (1/a^2)\partial p/\partial\xi = -(\rho/a^2)(u\partial u/\partial\xi + v\partial v/\partial\xi) \quad (7)$$

and using this equation to eliminate the density derivative, Eq. (3) becomes

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial u}{\partial \xi} - \frac{uv}{a^2}\frac{\partial v}{\partial \xi} + \frac{2\pi r L}{\dot{m}}\rho \times \left(u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta}\right) + \frac{vL}{r} = 0 \quad (8)$$

Eqs. (5) and (8) are to be solved for u and v where in terms of the total velocity and flow direction angle

$$u = V\cos\theta \quad \text{and} \quad v = V\sin\theta \quad (9)$$

The speed of sound is given by

$$a^2 = [(\gamma+1)/2]a^{*2}[1 - (\gamma-1)/(\gamma+1)V^{*2}] \quad (10)$$

and for isentropic flow along a streamline the pressure and density can be expressed in terms of the stagnation pressure,

p_s, where

$$p = p_s\{1 - [(\gamma - 1)/(\gamma + 1)]V^{*2}\}^{\gamma/(\gamma-1)} \quad (11)$$

and

$$\rho = [2/(\gamma + 1)]\gamma p_s/a^{*2}\{1 - [(\gamma - 1)/(\gamma + 1)]V^{*2}\}^{1/(\gamma-1)} \quad (12)$$

where $V^* = V/a^*$, V is the total velocity and $a^* = a^*(\eta)$ is the critical speed of sound and is related to the stagnation speed of sound by

$$a^{*2} = [2/(\gamma + 1)]a_t^2 \quad (13)$$

Using an order-of-magnitude analysis similar to that used by Hall, the appropriate scale length for the transonic region becomes

$$L = [(\gamma + 1)/2]^{1/2}\epsilon^{1/2}r_{t_o} \quad (14)$$

and the appropriate scaled flow direction angle is given by

$$\phi = \theta/\{[(\gamma + 1)/2]^{1/2}\epsilon^{1/2}\} \quad (15)$$

where $\epsilon = r_{t_o}/R$ and R is the radius of curvature of the wall contour at the throat. It is convenient, as will be apparent later, to define the reference radius r_{t_o} as the radius of the throat which will accommodate the equivalent one-dimensional mass flow. The quantity ϵ is small compared to unity and will be taken as a perturbation parameter.

Using Eqs. (9–15), the final forms of Eq. (5) and (8) become

$$(1 - V^{*2}) \cos\theta \frac{\partial V^*}{\partial \xi} - \epsilon\left(1 - \frac{\gamma - 1}{\gamma + 1} V^{*2}\right) \times \left\{\frac{\gamma + 1}{2} \frac{\sin\theta}{[(\gamma + 1)\epsilon/2]^{1/2}} V^* \frac{\partial \phi}{\partial \xi} - \frac{2\pi r_{t_o}^2}{\dot{m}} \frac{\gamma p_s}{a^*} \times \left(1 - \frac{\gamma - 1}{\gamma + 1} V^{*2}\right)^{1/(\gamma-1)} \zeta V^{*2} \frac{\partial \phi}{\partial \eta} - \frac{\gamma + 1}{2} \frac{V^*}{\zeta} \frac{\sin\theta}{[(\gamma + 1)\epsilon/2]^{1/2}}\right\} = 0 \quad (16)$$

and

$$\frac{\sin\theta}{[(\gamma + 1)\epsilon/2]^{1/2}} \frac{\partial V^*}{\partial \xi} + V^* \cos\theta \frac{\partial \phi}{\partial \xi} = \frac{2}{\gamma + 1} \frac{2\pi r_{t_o}^2}{\dot{m}} \times \zeta \frac{\gamma p_s}{a^*} \left[1 - \frac{\gamma - 1}{\gamma + 1} V^{*2}\right]^{1/(\gamma-1)} V^* \frac{\partial V^*}{\partial \eta} \quad (17)$$

where $\zeta = r/r_{t_o}$. It is not explicitly apparent in these equations that a^* depends upon the stream function.

The normalized radius appearing in Eqs. (16) and (17) must be determined as a function of ξ and η. This can be accomplished using the stream function relation $\rho u r = \partial\psi/\partial r$ to obtain

$$\zeta^2 = \left(\frac{\gamma + 1}{2}\right) \frac{\dot{m}}{\pi r_{t_o}^2} \times \int_0^\eta \frac{a^*(\eta)d\eta}{\gamma p_s V^* \cos\theta[1 - \gamma - 1/\gamma + 1\ V^{*2}]^{1/(\gamma-1)}} \quad (18)$$

The solution to Eqs. (16) and (17) may be carried out via series expansions of the form

$$V^* = 1 + \epsilon V_1^* + \epsilon^2 V_2^* + \ldots \quad (19)$$

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots \quad (20)$$

$$\zeta = \zeta_0 + \epsilon\zeta_1 + \epsilon^2\zeta_2 + \ldots \quad (21)$$

Introducing these series into Eq. (18) and equating coefficients of like powers of ϵ gives

$$\zeta_0 = \left[\frac{\dot{m}}{\gamma p_s \pi r_{t_o}^2} \left(\frac{\gamma + 1}{2}\right)^{\gamma/(\gamma-1)} \int_0^\eta a^*(\eta)d\eta\right]^{1/2} \quad (22)$$

$$\zeta_1 = 0 \quad (23)$$

$$\zeta_2 = \frac{1}{2\zeta_0}\left[\frac{\dot{m}}{\gamma p_s \pi r_{t_o}^2} \left(\frac{\gamma + 1}{2}\right)^{2\gamma-1/(\gamma-1)} \int_0^\eta a^* V_1^{*2} d\eta\right] \quad (24)$$

Introducing Eqs. (19–21) into Eq. (16) and equating coefficients of the ϵ^2 and ϵ^3 terms gives

$$2V_1^* \frac{\partial V_1^*}{\partial \xi} - \frac{2\pi r_{t_o}^2}{\dot{m}} \frac{\gamma p_s}{a^*} \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)} \times \zeta_0 \frac{\partial \phi_1}{\partial \eta} - \frac{1}{\zeta_0} \phi_1 = 0 \quad (25)$$

and

$$2V_1^* \frac{\partial V_2^*}{\partial \xi} + (V_1^{*2} + 2V_2^*) \frac{\partial V_1^*}{\partial \xi} + \phi_1 \frac{\partial \phi_1}{\partial \xi} - \frac{2\pi r_{t_o}^2}{\dot{m}} \frac{\gamma p_s}{a^*} \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)} \zeta_0\left[\frac{\partial \phi_2}{\partial \eta} + (2 - \gamma)V_1^* \frac{\partial \phi_1}{\partial \eta}\right] - \frac{1}{\zeta_0} [(2 - \gamma)V_1^*\phi_1 + \phi_2] = 0 \quad (26)$$

Equating coefficients of the ϵ and ϵ^2 terms in Eq. (17) gives

$$\frac{\partial \phi_1}{\partial \xi} = \frac{2\pi r_{t_o}^2}{\dot{m}} \frac{\gamma p_s}{a^*} \zeta_0 \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)} \frac{\partial V_1^*}{\partial \eta} \quad (27)$$

$$\phi_1 \frac{\partial V_1^*}{\partial \xi} + \frac{\partial \phi_2}{\partial \xi} + V_1^* \frac{\partial \phi_1}{\partial \xi} = \frac{2\pi r_{t_o}^2}{\dot{m}} \frac{\gamma p_s}{a^*} \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)} \zeta_0 \frac{\partial V_2^*}{\partial \eta} \quad (28)$$

Eqs. (25) and (27), when solved for V_1^* and θ_1 represent a first approximation to the exact solution. Eqs. (26) and (28) when solved for V_2^* and θ_2 together with the first approximation would give a second approximation to the solution.

The solutions to the above quations must satisfy certain boundary conditions on $\theta(\xi,\eta)$ such that the flow is compatible with the nozzle geometry. For axially symmetric flow, the center line flow direction angle is zero and, hence,

$$\phi_1(\xi,0) = \phi_2(\xi,0) = 0 \quad (29)$$

The outermost streamline must follow the wall contour given by

$$r_w/r_{t_o} = \zeta_w(x) = \zeta_w(0) + x^2/2Rr_{t_o} + \sigma x^4/8R^3r_{t_o} + \ldots \quad (30)$$

where $\sigma = 1$ for a circular contour, $\sigma = 0$ for a parabolic contour and $\sigma = -1$ for a hyperbolic contour. Differentiating the above equation to obtain $\tan\theta_w$, introducing dimensionless variables and expanding into a series, we obtain

$$\phi(\xi,\eta) = \xi\epsilon + \tfrac{1}{6}(3\sigma - 2)[(\gamma + 1)/2]\xi^3\epsilon^4 + \ldots \quad (31)$$

Hence, the conditions to be satisfied at the wall ($\eta = 1$) are

$$\phi_1(\xi,1) = \xi \quad \text{and} \quad \phi_2(\xi,1) = 0 \quad (32)$$

Introducing dimensionless variables, Eq. (30) can be written

$$\zeta_w = 1 + \{\zeta_{2w}(0) + [(\gamma + 1)/4]\xi^2\}\epsilon^2 + \ldots \quad (33)$$

With the expression for the wall contour in the above form we then take $\zeta_0 = 1$ at $\eta = 1$. This specification is consistent with the definition of r_{t_o}. When $\zeta_0 = 1$, $r = r_{t_o}$, and the total mass flow with the flowfield defined by the zeroth order solution ($\epsilon = 0$) requires a nozzle radius of r_{t_o}. The radius r_{t_o} is essentially determined by putting $\zeta_0 = 1$ and $\eta = 1$ in Eq. (22) to give

$$\frac{\dot{m}(\gamma + 1/2)^{\gamma/(\gamma-1)}}{\gamma p_s \pi r_{t_o}^2} \int_0^1 a^* d\eta = 1 \quad (34)$$

It is understood in the above discussion that $\dot{m}$ is to be interpreted as the actual mass flow rate through the duct. Then, as will be shown, various approximations to the solution will give rise to various approximations to the throat radius to accommodate that mass flow. This procedure is different from that used by Hall and others who keep the nozzle throat radius fixed and obtain various approximations to the mass flow. However, the use of the stream function as an independent variable suggests the procedure used here.

It would appear that before Eqs. (25) and (27) are solved for V_1^* and ϕ_1, the quantity $a^*(\eta)$ would have to be specified and that $\zeta_0(\eta)$ be determined from Eq. (22). Fortunately, this is not the case since η can be eliminated from Eqs. (25) and (27) with the $\zeta_0(\eta)$ relation given by Eq. (22) thus yielding

$$2V_1^*\partial V_1^*/\partial\xi - \partial\phi_1/\partial\zeta_0 - (1/\zeta_0)\phi_1 = 0 \quad (35)$$

and

$$\partial\phi_1/\partial\xi = \partial V_1^*/\partial\zeta_0 \quad (36)$$

where the boundary conditions on $\phi_1(\xi,\zeta_0)$ are $\phi_1(\xi,0) = 0$ and $\phi_1(\xi,1) = \xi$. These equations are similar to the first approximation equations of Hall [Eqs. (73) and (74) of Ref. 2] and their solution gives

$$V_1^* = \tfrac{1}{2}\zeta_0^2 - \tfrac{1}{4} + \xi \quad (37)$$

$$\phi_1 = \tfrac{1}{4}\zeta_0^3 - \tfrac{1}{4}\zeta_0 + \zeta_0\xi \quad (38)$$

Transforming from ξ,η to ξ,ζ_0 coordinates in Eqs. (26) and (28) with the aid of Eq. (22) gives for the second approximation

$$2V_1^*(\partial V_2^*/\partial\xi) + (2V_2^* + V_1^{*2})(\partial V_1^*/\partial\xi) + \phi_1\partial\phi_1/\partial\xi - \partial\phi_2/\partial\zeta_0 - (2-\gamma)V_1^*(\partial\phi_1/\partial\zeta_0) - 1/\zeta_0[\phi_2 + (2-\gamma)V_1^*\phi_1] = 0 \quad (39)$$

$$\phi_1\partial V_1^*/\partial\xi + \partial\phi_2/\partial\xi + V_1^*(\partial\phi_1/\partial\xi) = \partial V_2^*/\partial\zeta_0 \quad (40)$$

where the boundary conditions on $\phi_1(\xi,\zeta_0)$ are $\phi_2(\xi,0) = 0$ and $\phi_2(\xi,1) = 0$. Following the procedure used by Hall, the solution for the second approximation terms is found to be

$$V_2^* = \frac{2\gamma+9}{24}\zeta_0^4 - \frac{4\gamma+15}{24}\zeta_0^2 + \frac{10\gamma+57}{288} + \xi\left(\zeta_0^2 - \frac{5}{8}\right) - \frac{2\gamma-3}{6}\xi^2 \quad (41)$$

$$\phi_2 = \frac{8\gamma+15}{72}\zeta_0^5 - \frac{20\gamma+45}{96}\zeta_0^3 + \frac{28\gamma+75}{288}\zeta_0 + \left(\frac{4\gamma+9}{12}\right)(\zeta_0^3 - \zeta_0)\xi \quad (42)$$

This is similar in form to Hall's second solution [Eqs. (77) and (78) of Ref. 2]. To complete the second order solution, the second order approximation to the radius must be determined. Changing variables with the aid of Eq. (22), Eq. (24) can be written

$$\zeta_2 = \left(\frac{\gamma+1}{2}\right)\frac{1}{\zeta_0}\int_0^{\zeta_0} V_1^*\zeta_0 d\zeta_0 \quad (43)$$

Using Eq. (37) for V_1^* and performing the integration in the above equation, Eq. (21) for the radial coordinate becomes

$$\zeta = \zeta_0 + [(\gamma+1)/6]1/\zeta_0[(\xi - \tfrac{1}{4} + \tfrac{1}{2}\zeta_0^2)^3 - (\xi - \tfrac{1}{4})^3]\epsilon^2 + \ldots \quad (44)$$

which at the wall ($\zeta_0 = 1$) becomes

$$\zeta_w = 1 + \{(\gamma+1)/192 + [(\gamma+1)/4]\xi^2\}\epsilon^2 + \ldots \quad (45)$$

It is seen that this equation is in accord with Eq. (33). In situations where the second order solution adequately describes the solution the throat radius would be taken as

$$r_t = \{1 + [(\gamma+1)/192]\epsilon^2\}r_{t_0} \quad (46)$$

Conclusions

A first and second order solution of the transonic equations for nonuniform total energy has been obtained using the stream function as an independent variable. Neither the first solution expressed by Eqs. (37) and (38) nor the second solution expressed by Eqs. (41) and (42) is explicitly dependent upon $a^*(\eta)$. Hence, the dimensionless quantities V^*, ϕ and the sonic line shape are formally independent of any existing variation in the total temperature. However, the details of the rotational flowfield such as the distributions of a, ρ and V are dependent upon $a^*(\eta)$.

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Probe Geometry Effects on Turbulent Plasma Diagnostics

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In recent years, the cylindrical electrostatic probes have been most extensively used for turbulent plasma diagnostics.[1–6] The object is to determine the statistics of electron density fluctuations in turbulent plasmas with high-spatial resolution. In this Note, we report some studies on the perturbing effects of the probe geometry on the turbulent plasma parameters when they are used to measure the local statistical properties of turbulent plasma flows. The effects studied are: 1) frequency filtering effects of the cylindrical electrostatic probes on the power-frequency spectrum when placed axially along the turbulent plasma flow. 2) The "wake" effect of an upstream probe on a downstream probe when the pair is used to measure various statistical moments of the turbulent plasma flow.

A vortex-stabilized argon arc (100 amp at 20 v) has been used as a plasma source[4] which opens into an evaucated test section (3.5 ft diam, 5 ft long) through a nozzle. The typical operating pressure in the arc chamber is 150 torr and the test section pressure is 3 torr. The plasma jet flows into the test section as an extended plume which can be run into the turbulent mode by varying the gas flow rate.

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